UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE J-2 209

TITLE — Roots of a Polynomial (DOI only)

TYPE — Entire program

ACCURACY — Depends on condition of the polynomial. Usually about 9 decimal places.

DURATION — $0.082\, n^2 + 5.2\, n + 18$ seconds

The time will depend partly on the distribution of roots.

DESCRIPTION — This program calculates the roots of a polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + ---- + a_n = 0$$

whose coefficients have been punched on tape. An n'th degree polynomial having complex coefficients may be punched and n complex roots obtained. Real numbers are treated as complex numbers whose imaginary parts are zero.

METHOD OF USE — The master tape is read into the computer and then the tape containing the coefficients is read. As each root is computed it will be punched. After all n roots have been punched the computer will stop on a black switch stop* and another tape of coefficients may be read. Hence one may prepare several equations on a single tape and have them all solved by leaving the black switch in the disable position. If this is done the tape should be terminated with the symbol N. When this symbol is read in place of a new set of coefficients the routine will stop on OF.

PREPARING THE TAPE OF COEFFICIENTS — To prepare a tape of coefficients of the polynomial

$$a_0 x^n + a_1 x^{n-1} + --- + a_n = 0$$

one should first type n as a decimal integer followed by a carriage return. The n + 1 coefficients are now typed in order a_0, a_1, ---, a_n. Each must be followed by a carriage return. Each coefficient a_j must be regarded as a complex number in floating decimal form

$$a_j = (A + iB) \times 10^p$$

where $|A| < 1$ and $|B| < 1$ and p is an integer.

The three numbers A, B, and p are typed in order, A and B as signed fractions with the decimal point preceding the first digit in each case, and p as a signed integer. The absence of digits after a sign is treated as zero.

EXAMPLE The equation

$$x^5 + 24x^4 + (3 - 64i)\,x^3 - (.05 + .0034i)\,x^2 + .39 = 0$$

would be prepared as

5
+1++1
+24++2
+03-64+2
-5-034-1
+++
+39++

FORM OF THE RESULTS The roots are printed in floating decimal form A B p, with A and B printed as signed fractions to nine decimal places and p printed as a signed integer. Each root is followed by a residual printed to 3 decimal places which is obtained by substituting the root back into the original equation. The residuals also appear in floating decimal form.

MATHEMATICAL METHOD The program uses a 1.8th order iterative procedure to find a root. Each successive iterant is obtained by solving for the nearer root of the quadratic passing through the last three iterants. This solution is accomplished by a variation of the standard quadratic formula. By use of this method the number of evaluations of the function is reduced, and hence the time to solve equations of high degree. As each root is found it is divided into the polynomial, thus reducing its degree by one, and the process is repeated.

The specific algorithm which is used may be described as follows: Given quantities f_0, f_1, f_2, x, λ, δ, h we calculate λ' by the formula

$$\lambda' = \frac{-2f_2\,\delta}{b \pm \sqrt{b^2 - 4f_2\,\delta\,\lambda\,(f_0\,\lambda - f_1\,\delta + f_2)}}$$

where $b = f_0\,\lambda^2 - f_1\,\delta^2 + f_2\,(\lambda + \delta)$

The sign before the radical is chosen so as to make the denominator have the larger magnitude. In one iteration we replace

λ by λ'

h by $\lambda' h$

x by $x + \lambda' h$

δ by $1 + \lambda'$

f_2 by $f(x + \lambda' h)$

f_1 by f_2

f_0 by f_1

Before calculating each root we start with the initial values.

$f_0 = f(0) - f'(0) + 1/2\, f''(0)$

$f_1 = f(0) + f'(0) + 1/2\, f''(0)$

$f_2 = f(0)$

$x = 0$

$\lambda = -1/2$

$\delta = +1/2$

$h = -1$

We make use of the relations

$f(0) = a_n$, $f'(0) = a_{n-1}$, $1/2\, f''(0) = a_{n-2}$

A final value of x, say x_i, is printed when

$\dfrac{|x_i - x_{i-1}|}{|x_i|} < 10^{-9}$, that is when the change in x is no more than one part in 10^9.

In order to ensure convergence a special process is used whenever $f_2/f_1 > 10$. In this case λ' is replaced by $1/2\ \lambda'$ and h, x, and f_2 are recomputed. The original process is then resumed. Final convergence is not affected by this process.

DATE April 19, 1956 R 7/20/59
PROGRAMMED BY D. E. Muller
APPROVED BY J. P. Nash

DEM/mge

LOCATION	ORDER	NOTES
Library Routine X-1		Decimal Order Input
	00 3K	
0	00 F	
	00 120F	
1	00 F	
	00 123F	
2	00 F	Preset Parameters
	00 373F	
3	00 F	
	00 698F	
4	00 F	
	00 93F	
	00 371K	
0	00 F	
	00 F	$a_{-1} = 0$
1	00 F	
	00 F	
	00 120K	
0	00 F	
	00 F	
1	00 F	
	00 F	Floating accumulator
2	00 F	
	00 F	
	00 123K	
Library Routine A-5		**Complex Number Arithmetic**
	00 8K	
0	50 79L	
	81 4F	
1	L0 203S4	
	32 77L	**Read n**
2	L4 203S4	
	74 203S4	

LOCATION	ORDER	NOTES	PAGE 2
3	00 4F		
	91 4F		
4	32 2L		
	K5 F		
5	42 8L	Plant n	
	42 46L		
6	S5 F		
	42 11L		
7	42 70L		
	50 7L	Enter **A-5**	
8	26 S4		
	0K F		
9	88 F	**Read polynomial**	
	0S S5		
10	0S S6		
	0[illegible] 9L		
11	8F 2F		
	2K F		
12	8[illegible] 62L		
	05 1022S5		
13	8S 4S7		
	04 1018S5		
14	04 1020S5		
	8S 2S7	Form f_0, f_1, f_2	
15	00 1020S5	**initially**	
	00 1020S5		
16	8S S7		
	85 78L		
[illegible]	0K 4F		
	0S 6S7	**Form initial δ, x, λ, h**	
18	30 6S7		
	02 17L		
19	85 S7		
	87 10S7		

LOCATION	ORDER	NOTES
20	8S 14S7	
	87 10S7	
21	8[illegible] 16S7	
	81 2S7	
22	87 6S7	
	8S 18S7	
23	87 6S7	
	8S 20S7	
24	85 10S7	
	84 6S7	
25	87 4S7	
	84 16S7	
26	84 20S7	
	8S 16S7	
27	87 16S7	
	8S 20S7	Compute λ
28	8K 2F	by the formula
	87 4S7	
29	87 6S7	
	8S 22S7	
30	81 14S7	
	80 18S7	
31	80 4S7	
	87 10S7	
32	87 22S7	
	8S 14S7	
33	84 14S7	
	84 20S7	
34	8J 225S4	
	8S 14S7	
35	8J 42S4	
	87 16S7	
36	82 37L	
	81 14S7	

LOCATION	ORDER	NOTES PAGE 4
37	8S 14S7	
	85 16S7	
38	84 14S7	
	8S 14S7	
39	81 22S7	
	86 14S7	
40	8S 10S7	
	87 12S7	
41	8S 12S7	
	84 8S7	
42	8S 8S7	
	8N 64L	Form new h, x, δ, f_0, f_1
43	8K 1F	
	84 10S7	
44	8S 6S7	
	85 2S7	
45	8S S7	
	85 4S7	
46	8S 2S7	
	0K F	
47	85 S5	
	87 8S7	
48	04 2S5	Form new f_2
	02 47L	
49	8S 4S7	
	8J 57L	
50	35 10S7	
	87 78L	
51	8S 10S7	
	85 12S7	Replace λ' by 1/2 λ'
52	87 78L	and recompute h , x, and δ
	8S 12S7	
53	85 8S7	
	80 12S7	

LOCATION	ORDER	NOTES
54	8S 8S7	
	8K 1F	
55	84 10S7	
	8S 6S7	
56	8K 1F	
	82 46L	
57	L5 2S7	
	50 3S7	
58	10 5F	
	0[illegible] 10F	Test for $f_2/f_1 > 10$
59	L0 1[illegible]	
	F4 1[illegible]8S4	
60	[illegible]6 29S4	
	[illegible]5 81L	
61	46 2S4	
	26 29S4	
62	L5 46L	
	F[illegible] 79L	
63	42 46L	
	42 72L	
64	28 29S4	
	L[illegible] 12S7	
65	[illegible] 13S7	
	[illegible] 5F	
66	0[illegible] 10F	Test for convergence
	L0 1S3	
67	[illegible] 80L	
	36 29S4	
68	22 68L	Enter A-5
	50 68L	
69	26 S4	
	89 9F	Print root
70	85 56	
	1K F	

LOCATION	ORDER	NOTES
71	87 867	
	14 286	Form residual and print
72	13 71L	
	OK F	
73	89 3F	
	05 85	
74	87 887	
	04 285	Divide through by root
75	08 285	
	02 73L	
76	23 12L	
	8J 77L	Test for end
77	24 L	
	OF F	
78	20 F	
	00 16F	1/2 in floating form
79	00 F	
	00 F	
80	00 F	Convergence criterion
	00 8F	
81	00 18L	
	00 F	
Library Routine X-7		Sum Check
	24 8N	